Homogeneous, expanding universes 2

Block Course Introduction to Astronomy and Astrophysics

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What we did in part 1

- Model set-up for isotropic, homogeneous universe
- Scale-factor expansion
- Light propagation in an expanding universe
- Cosmological redshift
- How densities scale with $a(t)$
- Friedmann equations (first and second order)

Next: How to solve

$$\frac{\dot{a}^2 + Kc^2}{a^2} = \frac{8\pi G}{3}\rho \quad \text{and} \quad \dot{\rho} = -3(\rho + p/c^2)\frac{\dot{a}}{a}$$

Revisiting three kinds of content (matter, radiation, dark energy):

Name	index	scaling behavior
matter	m	$\rho_m(t) \sim a(t)^{-3}$
radiation	r	$\rho_r(t) \sim a(t)^{-4}$
dark energy	Λ	$\rho_\Lambda(t) \sim \text{const.}$

Assume non-interaction ⇔ total density sum of partial densities:

$$\rho = \rho_m + \rho_r + \rho_\Lambda$$

Considerable simplification: make Friedmann equation dimensionless by introducing critical density

$$\rho_c \equiv \frac{3H_0^2}{8\pi G}$$

Omega parameters

Scaling behavior from current value, at time t_0, rescale using critical density:

$$
\begin{aligned}
\rho_m(t) &= \underbrace{\Omega_m \cdot \rho_c}_{\rho_m(t_0)} \cdot \left(\frac{a(t)}{a(t_0)}\right)^{-3} \\
\rho_r(t) &= \underbrace{\Omega_r \cdot \rho_c}_{\rho_r(t_0)} \cdot \left(\frac{a(t)}{a(t_0)}\right)^{-4} \\
\rho_\Lambda(t) &= \Omega_\Lambda \cdot \rho_c \\
Kc^2 = \frac{kc^2}{R_0^2} &= -\Omega_K \cdot [a(t_0)\, H_0]^2
\end{aligned}
$$

Dimensionless Friedmann equation

$$\text{With} \quad x(t) \equiv \frac{a(t)}{a(t_0)} = \frac{1}{1+z} \quad \text{and} \quad H(t) \equiv \frac{\dot{a}(t)}{a(t)}.$$

first-order Friedmann equation

$$\frac{\dot{a}^2 + Kc^2}{a^2} = \frac{8\pi G}{3}\rho$$

becomes

$$H(t)^2 = H_0^2 \left[\Omega_\Lambda + \Omega_K\, x^{-2} + \Omega_m\, x^{-3} + \Omega_r\, x^{-4}\right]$$

or

$$\int \mathrm{d}t = \int \frac{\mathrm{d}x}{H_0\, x\, \sqrt{\Omega_\Lambda + \Omega_K\, x^{-2} + \Omega_m\, x^{-3} + \Omega_r\, x^{-4}}}.$$

Simple integral – call in the mathematicians!

Eliminating Ω_K if necessary

Present-time Friedmann equation: $x = 1$, so

$$\Omega_\Lambda + \Omega_K + \Omega_m + \Omega_r = 1$$

Consequences: Omegas not independent, we can eliminate Ω_K

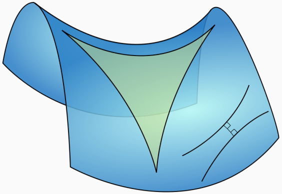
Rewrite Ω_K in terms of k, with $\Omega \equiv \Omega_\Lambda + \Omega_m + \Omega_r$:

$$k = \left(\frac{a(t_0)\, H_0\, R_0}{c}\right)^2 (\Omega - 1)$$

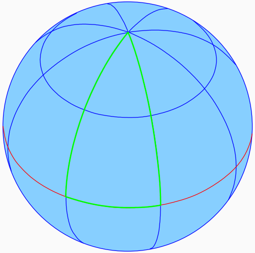
Evidently, the sign k is related to the ratio Ω of total and critical density.
But what does it mean?

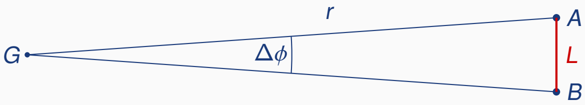
What our simplification misses

When derived from the Einstein field equations, k has a specific geometric meaning: at fixed time t,
with curvature radius $R_0 \equiv c/(H_0\, a_0 \sqrt{|\Omega_K|})$,

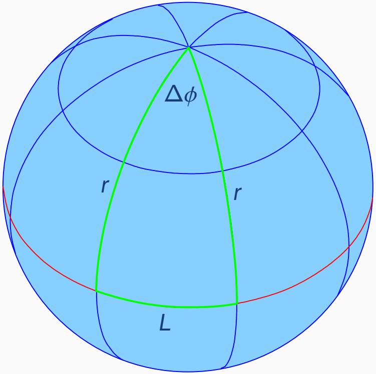


$k = +1$	spherical space:	$L = a(t)\, R_0 \sin\left[\frac{r}{R_0\, a(t)}\right] \cdot \Delta\phi$
$k = 0$	Euclidean space:	$L = r \cdot \Delta\phi$
$k = -1$	hyperbolical space:	$L = a(t)\, R_0 \sinh\left[\frac{r}{R_0\, a(t)}\right] \cdot \Delta\phi$



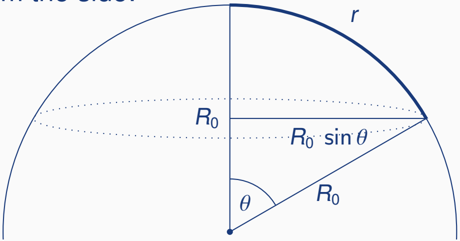


with r, L physical distances



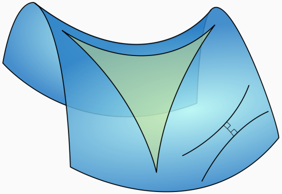
r is part of a great circle on the spherical surface.

View from the side:



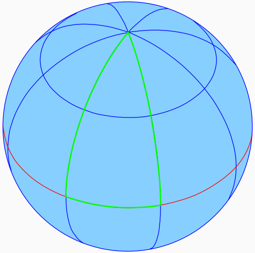
For θ in radians: $r = R_0 \cdot \theta$,
while with this setup $L = R_0 \sin\theta \, \Delta\phi$, so

$$L = R_0 \sin\left[\frac{r}{R_0}\right] \cdot \Delta\phi$$



Scaled by the critical density, mass density determines geometry:

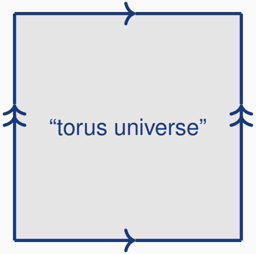
$$\begin{array}{llll} \Omega > 1 & \Leftrightarrow & k = +1 & \text{spherical space} \\ \Omega = 1 & \Leftrightarrow & k = 0 & \text{Euclidean space} \\ \Omega < 1 & \Leftrightarrow & k = -1 & \text{hyperbolical space} \end{array}$$



$\Omega > 1$	$\Leftrightarrow$	spherical,	finite,	cosmos will collapse
$\Omega = 1$	$\Leftrightarrow$	Euclidean,	infinite,	cosmos will keep expanding
$\Omega < 1$	$\Leftrightarrow$	hyperbolical,	infinite,	cosmos will keep expanding

Synonyms in older texts: finite = "closed universe", infinite = "open universe"

- Ω controls local geometry only
- local geometry does not determine topology
- direct prediction for collapse or not only for $\Lambda = 0$



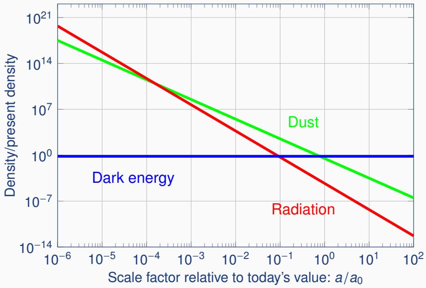
- $k = 0$: 10 finite, 8 infinite homogeneous spaces (Riazuelo et al. 2004)
- $k = +1$: countably infinitely many homogeneous spaces, all finite (Gausmann et al. 2001)
- $k = -1$: uncountably inifinitely many, some finite, some infinite (Cornish et al. 1998)

Second-order Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p/c^2) = -\frac{4\pi G}{3}(\rho_m + 2\,\rho_r - 2\,\rho_\Lambda) = -\frac{H_0^2}{2}\left[\frac{\Omega_m}{x^3} + \frac{2\Omega_r}{x^4} - 2\Omega_\Lambda\right]$$

In our model, only Ω_Λ can drive (positively) accelerated expansion — not as "negative mass', but via pressure!

For a universe that started small: initial momentum + deceleration can carry into acceleration phase



Re-collapse possible?

First-order Friedmann equation (let's ignore Ω_r, think big scales):

$$H(t)^2 = H_0^2\left[\Omega_\Lambda + \Omega_K\,x^{-2} + \Omega_m\,x^{-3}\right]$$

Initial expansion followed by recollapse $\Rightarrow H(t_{tp}) = 0$ for some turning-point time t_{tp}

Turning point possible means

$$\Omega_\Lambda + \Omega_K\,x^{-2} + \Omega_m\,x^{-3} = 0 \quad \Rightarrow \quad \Omega_\Lambda x^3 + \Omega_K\,x + \Omega_m = 0$$

For $\Omega_\Lambda = 0$: Turning-point scale factor is

$$x_{tp} = \frac{\Omega_m}{\Omega_m - 1}$$

and $x_{tp} > 0$ requires $\Omega_m > 1$, density larger than critical density. $\Omega_\Lambda \neq 0$ more complicated.

Second-order Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{H_0^2}{2}\left[\frac{\Omega_m}{x^3} + \frac{2\Omega_r}{x^4} - 2\Omega_\Lambda\right]$$

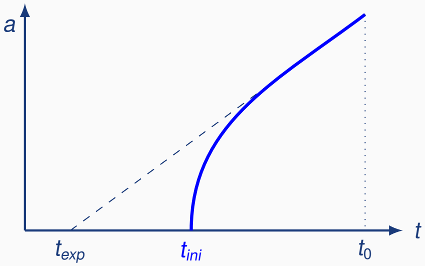
For expanding universes where Λ does not dominate completely, $\ddot{a}/a \leq 0$.

$$H(t)^2 = H_0^2\left[\Omega_\Lambda + \Omega_K\, x^{-2} + \Omega_m\, x^{-3} + \Omega_r\, x^{-4}\right]$$

shows $H(t)$ grows $\Rightarrow a(t)$ ever steeper in the past

Curve ends at some $a(t_{ini}) = 0$:
initial (Big Bang) singularity

Special case Hawking-Penroses singularity theorem



(with $t_{exp} = t_0 - 1/H_0$)

$$t = \int_0^t \mathrm{d}t = \int_0^x \frac{\mathrm{d}x'}{H_0\, x' \sqrt{\Omega_\Lambda + \Omega_K\,(x')^{-2} + \Omega_m\,(x')^{-3} + \Omega_r\,(x')^{-4}}}.$$

At small scales $x \ll 1$, radiation term dominates — neglect all other terms:

$$t = \int_0^t \mathrm{d}t = \int_0^x \frac{\mathrm{d}x'}{H_0\, x' \sqrt{\Omega_r\,(x')^{-4}}} = \frac{1}{H_0\sqrt{\Omega_r}} \int_0^x x'\,\mathrm{d}x' = \frac{x^2}{2H_0\sqrt{\Omega_r}}$$

Evolution of scale factor (flat radiation-only would be $\Omega_r = 1$):

$$a = a_0 \sqrt{2H_0\sqrt{\Omega_r}\, t} \propto \sqrt{t}.$$

Age of the universe if this were the only contribution until the present (it's not):

$$t_0 = \frac{1}{2H_0\Omega_r}$$

$$t = \int_0^t \mathrm{d}t = \int_0^x \frac{\mathrm{d}x'}{H_0\, x' \, \sqrt{\Omega_\Lambda + \Omega_K \,(x')^{-2} + \Omega_m \,(x')^{-3} + \Omega_r \,(x')^{-4}}}.$$

Flatness and matter-only means $\Omega_m = 1$, all other omegas zero:

$$H_0 \int_0^t \mathrm{d}t = \int_0^x \sqrt{x'} \, \mathrm{d}x'$$

so that

$$a = a_0 \left(\frac{3}{2} \cdot H_0 \; t \right)^{2/3} \propto t^{2/3}$$

Age of the universe:

$$t_0 = \frac{2}{3 H_0}$$

$$t - t_0 = \int_{t_0}^{t} \mathrm{d}t = \int_{1}^{x} \frac{\mathrm{d}x'}{H_0\, x' \sqrt{\Omega_\Lambda + \Omega_K\,(x')^{-2} + \Omega_m\,(x')^{-3} + \Omega_r\,(x')^{-4}}}.$$

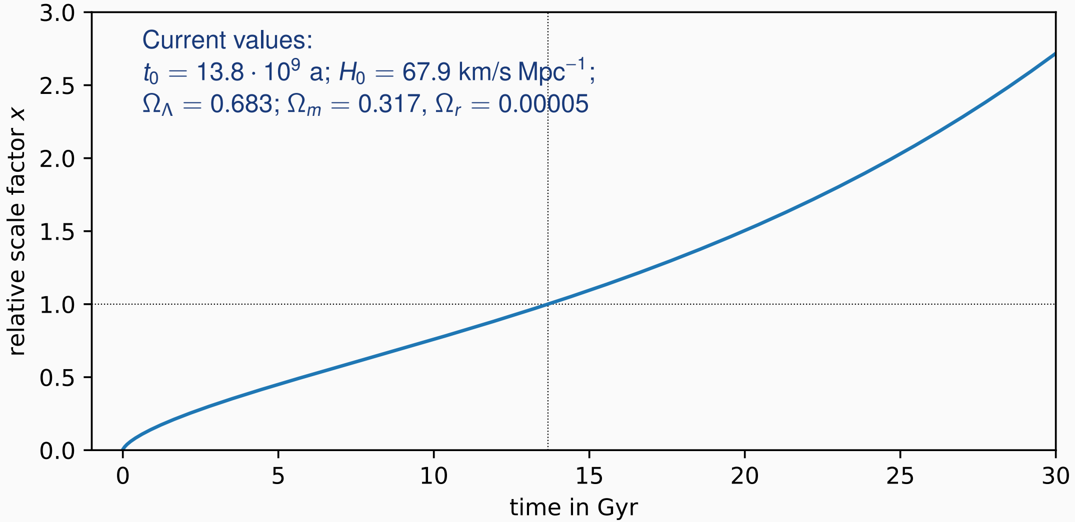
Only non-zero contribution Ω_Λ; flat universe $\Rightarrow \Omega_\Lambda = 1$.
To avoid divergence: integrate from present-day t_0, $x = 1$:

$$H_0 \cdot (t - t_0) = H_0 \int_{t_0}^{t} \mathrm{d}t = \int_{1}^{x} \frac{\mathrm{d}x}{x} = \ln(x)$$

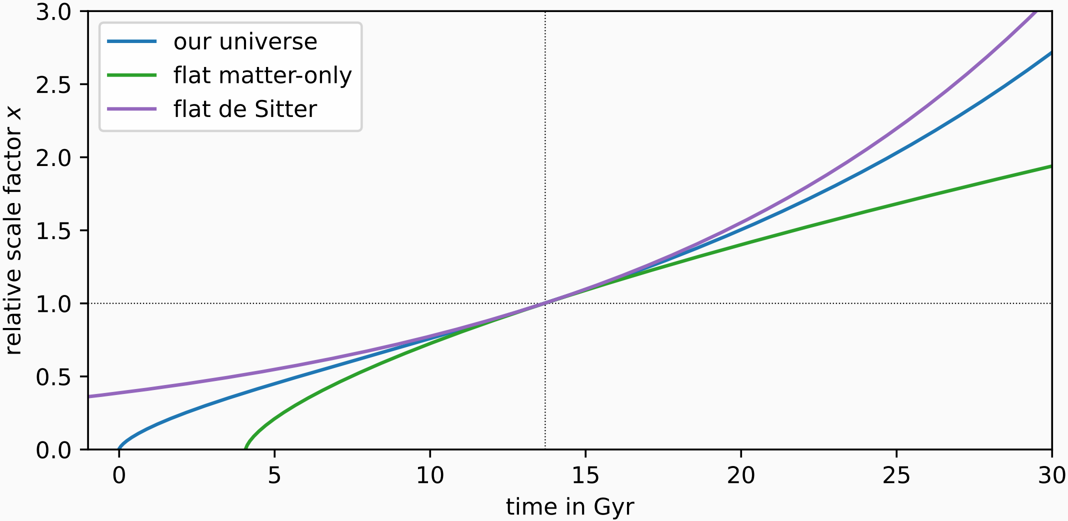
so that

$$a = a_0 \exp(H_0[t - t_0])$$

Infinitely old universe — no initial time with $a(t_{ini}) = 0$!


Current values:
$t_0 = 13.8 \cdot 10^9$ a; $H_0 = 67.9$ km/s Mpc^{-1};
$\Omega_\Lambda = 0.683$; $\Omega_m = 0.317$, $\Omega_r = 0.00005$
relative scale factor x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
25
30
time in Gyr

Our own universe vs. approximations fitted at t_0



- just quote the redshift z, very practical and directly observable
- comoving distances, good for tagging galaxies
- proper distances = comoving distance, if at time $t \neq t_0$ rescaled with scale factor (what I called "physical distances" above)
- lookback time $\equiv$ light travel time (light-years!)
- distant objects get dimmer $\Rightarrow$ luminosity distance d_L
- distant objects appear smaller $\Rightarrow$ angular distance d_A

Next: let's try to express all these distances in terms of redshift z!

From the Friedmann equation, we had the lookback time equation

$$t - t_0 = \int_{t_0}^{t} \mathrm{d}t = \int_{1}^{x} \frac{\mathrm{d}x'}{H_0\, x' \sqrt{\Omega_\Lambda + \Omega_K\,(x')^{-2} + \Omega_m\,(x')^{-3} + \Omega_r\,(x')^{-4}}}.$$

Obtain light-travel-time distance $d_{ltt} = c(t_0 - t)$, with $t_0 - t$ lookback time:

$$\text{Change of variables} \quad x = \frac{a(t)}{a(t_0)} = \frac{1}{1+z} \qquad \Rightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}z} = -\frac{1}{(1+z)^2}$$

$$\Rightarrow \quad d_{ltt} \equiv c(t_0 - t) = \frac{c}{H_0} \int_0^z \frac{\mathrm{d}z'}{(1+z')E(z')}$$

$$\text{with } E(z) = \sqrt{\Omega_\Lambda + \Omega_K\,(1+z)^2 + \Omega_m\,(1+z)^3 + \Omega_r\,(1+z)^4}$$

We had seen that

$$d_{comov} = c\, a(t_0) \int_{t_e}^{t_0} \frac{\mathrm{d}t}{a(t)}.$$

To rewrite in terms of z at emission time t_e, we make several changes of variable:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = H(t)a(t) \quad \Rightarrow \quad d_{comov} = c\, a(t_0) \int_{a(t_e)}^{a(t_0)} \frac{\mathrm{d}a}{a^2 H(a)}$$

$$x = \frac{a(t)}{a(t_0)} \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}a} = \frac{1}{a(t_0)} \quad \Rightarrow \quad d_{comov} = c \int_{x}^{1} \frac{\mathrm{d}x}{x^2 H(x)}$$

Comoving distance (evaluated at $t = t_0$) in terms of redshift z 2

Friedmann equation $H(x) = H_0 \cdot E(x)$, with $E(x) \equiv \sqrt{\Omega_\Lambda + \Omega_K\, x^{-2} + \Omega_m\, x^{-3} + \Omega_r\, x^{-4}}$, so

$$d_{comov} = \frac{c}{H_0} \int_x^1 \frac{\mathrm{d}x}{x^2\, E(x)}$$

Change of variable from x to z:

$$x = \frac{1}{1+z} \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}z} = -\frac{1}{(1+z)^2}$$

brings us to

$$d_{comov} = \frac{c}{H_0} \int_0^z \frac{\mathrm{d}z'}{E(z')}$$

Particle horizon = comoving distance light can have travelled since the Big Bang:

$$d_{comov,ph} = \frac{c}{H_0} \int_0^{\infty} \frac{\mathrm{d}z'}{E(z')} \quad \text{with} \quad E(z) = \sqrt{\Omega_\Lambda + \Omega_K (1+z)^2 + \Omega_m (1+z)^3 + \Omega_r (1+z)^4}$$

Matter-only universe (radiation-dominated is not better):

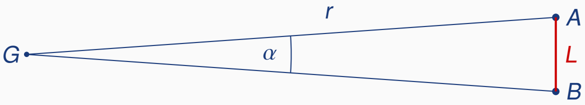
$$d_{comov,ph} = \text{ finite (see today's exercise!)}$$

Dark-energy-dominated-universe:

$$d_{comov,ph} = \infty$$

By giving our own universe an early exponential phase, we can increase $d_{comov,ph}$!

Consider Hubble-flow object transversally separated by L:



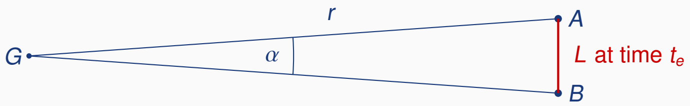
Present-time values define transversal proper distance $d_\perp$ at present time t_0 as:

$$L = d_\perp \cdot \alpha$$

From the earlier statement about spatial geometry, evaluated at $t = t_0$, we know

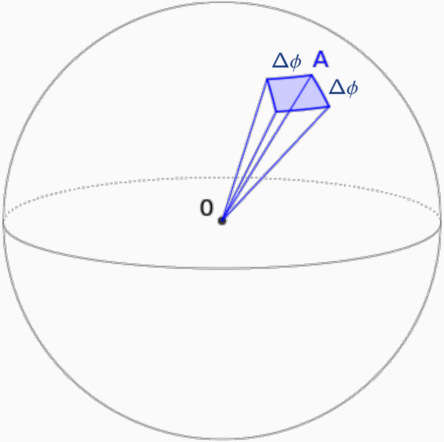
$$d_\perp = \begin{cases} \frac{c}{H_0\sqrt{|\Omega_K|}} \sin\left[\frac{d_{comov}}{c/(H_0\sqrt{|\Omega_K|})}\right] & \text{for } k = +1 \\ d_{comov} & \text{for } k = 0 \\ \frac{c}{H_0\sqrt{|\Omega_K|}} \sinh\left[\frac{d_{comov}}{c/(H_0\sqrt{|\Omega_K|})}\right] & \text{for } k = -1 \end{cases}$$

Like transversal-comoving, but rescaled so we obtain the distance L between two Hubble-flow objects at light emission time t_e, corresponding to cosmic redshift z:



Call this angular diameter distance d_A; from the previous formula for d_{comov}, we simply need to rescale L by $1/(1+z)$ to obtain the distance at time t_e:

$$d_A = \frac{1}{(1+z)} \frac{c}{H_0\sqrt{|\Omega_K|}} S_k\left[\sqrt{|\Omega_K|}\int_0^z \frac{\mathrm{d}z'}{E(z')}\right] \quad \text{where} \quad S_k = \begin{cases} \sin & \text{for } k=+1 \\ \mathrm{id} & \text{for } k=0 \\ \sinh & \text{for } k=-1 \end{cases}$$



Small square patch light from object at O in solid angle $\Delta\phi^2$ spreads out over detector area

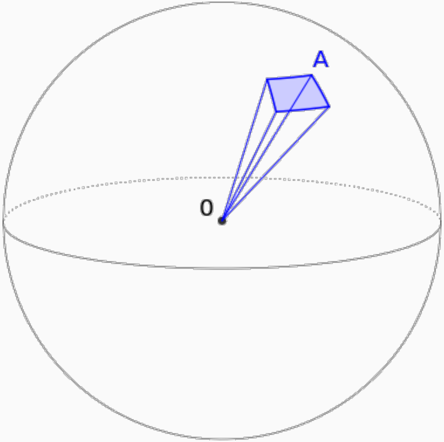
$$A = d_\perp{}^2 \cdot \Delta\phi^2$$

Initial energy output: luminosity L, energy per second, proportion of photons arriving at A is

$$\frac{\Delta\phi^2}{4\pi} = \frac{A}{4\pi d_\perp{}^2}$$

Two additional effects:

- photon energy redshifted, $E \mapsto E/(1+z)$
- Photon arrival rate time-dilated, additional factor $1/(1+z)$



Bolometric flux for observer:

$$F_{bol} = \frac{L}{4\pi d_\perp^2 (1+z)^2}.$$

define luminosity distance

$$d_L \equiv d_\perp (1+z) = d_A (1+z)^2$$

so that

$$F_{bol} = \frac{L}{4\pi d_L^2}.$$

Cosmological distances: summary

Light-travel time distance

$$d_{ltt} \equiv c(t_0 - t) = \frac{c}{H_0} \int_0^z \frac{\mathrm{d}z'}{(1+z')E(z')}$$

Comoving radial distance

$$d_{comov} = \frac{c}{H_0} \int_0^z \frac{\mathrm{d}z'}{E(z')}$$

Comoving transversal distance

$$d_\perp = \frac{c}{H_0\sqrt{|\Omega_K|}} S_k\left[\sqrt{|\Omega_K|}\int_0^z \frac{\mathrm{d}z'}{E(z')}\right] \quad \text{where} \quad S_k = \begin{cases} \sin & \text{for } k = +1 \\ \mathrm{id} & \text{for } k = 0 \\ \sinh & \text{for } k = -1 \end{cases}$$

Angular diameter distance

$$d_A = \frac{d_\perp}{1+z}$$

Luminosity distance

$$d_L = (1+z)d_\perp = (1+z)^2 d_A$$

with $E(z) = \sqrt{\Omega_\Lambda + \Omega_K (1+z)^2 + \Omega_m (1+z)^3 + \Omega_r (1+z)^4}$

Use Ned Wright's Javascript cosmology calculator (everyone else does!):

Enter values, hit a button

69.6 H_o
0.286 $Omega_M$
3 z
Open Flat
0.714 $Omega_{vac}$
General

Open sets $Omega_{vac} = 0$ giving an open Universe [if you entered $Omega_M < 1$]
Flat sets $Omega_{vac} = 1 - Omega_M$ giving a flat Universe.
General uses the $Omega_{vac}$ that you entered.
Source for the default parameters.

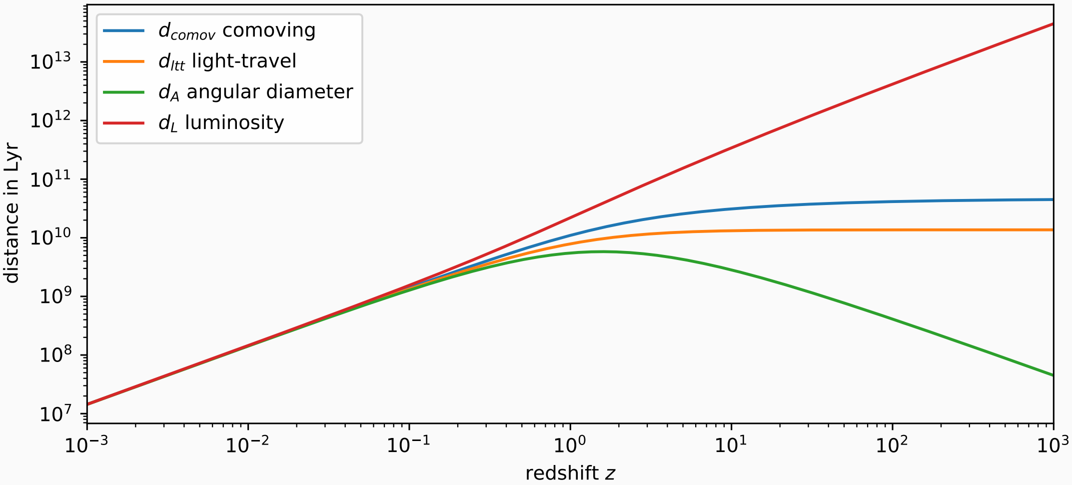
For $H_o = 69.6$, $Omega_M = 0.286$, $Omega_{vac} = 0.714$, $z = 3.000$

- It is now 13.721 Gyr since the Big Bang.
- The age at redshift z was 2.171 Gyr.
- The light travel time was 11.549 Gyr.
- The comoving radial distance, which goes into Hubble's law, is 6481.3 Mpc or 21.139 Gly.
- The comoving volume within redshift z is 1140.389 Gpc^3.
- The angular size distance D_A is 1620.3 Mpc or 5.2846 Gly.
- This gives a scale of 7.855 kpc/".
- The luminosity distance D_L is 25924.3 Mpc or 84.554 Gly.

1 Gly = 1,000,000,000 light years or $9.461*10^{26}$ cm.
1 Gyr = 1,000,000,000 years.
1 Mpc = 1,000,000 parsecs = $3.08568*10^{24}$ cm, or 3,261,566 light years.

https://www.astro.ucla.edu/ wright/CosmoCalc.html — cf. Wright, PASP, 118, 1711 (2006)

Different notions of distance



Distance comparisons for the parameters of our own universe

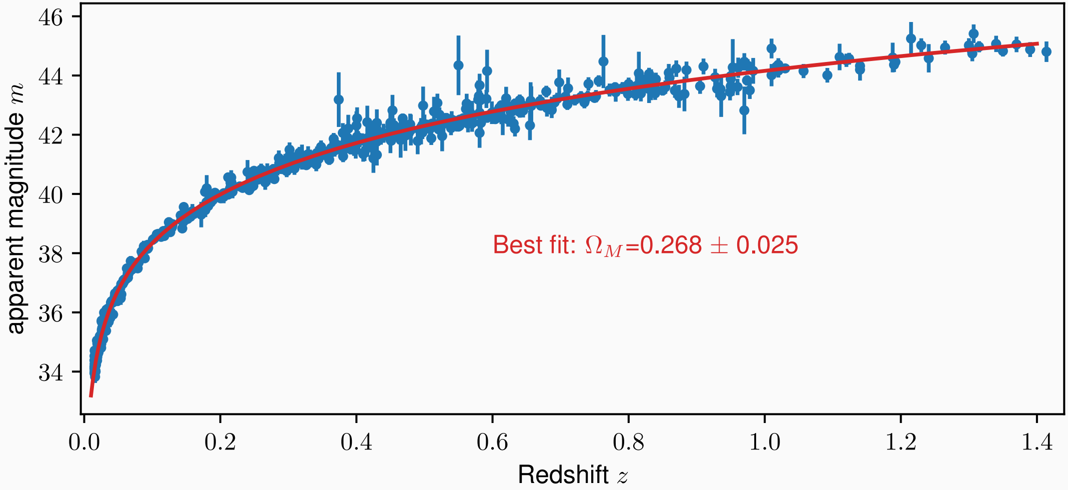
Assuming we know (from balloon experiments) that $k = 0$, flat universe,
we neglect radiation $\Rightarrow \Omega_\Lambda = 1 - \Omega_m$.

Simplest standard candle: absolute magnitude $M = const$.

Distance modulus for flux $\Rightarrow$ use luminosity distance d_L to obtain apparent brightness m:

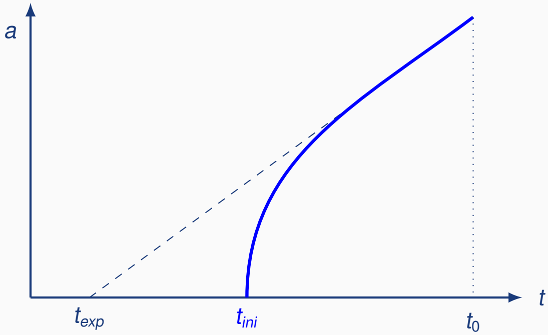
$$
\begin{aligned}
m(z) &= M + 5\log_{10}\left[\frac{d_L(z)}{10\,\mathrm{pc}}\right] = M + 5\log_{10}\left[\frac{c}{H_0 \cdot 10\,\mathrm{pc}}(1+z)\int_0^z \frac{\mathrm{d}z'}{E(z')}\right] \\
&= 5\log_{10}\left[(1+z)\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_m(1+z')^3 + 1 - \Omega_m}}\right] + C
\end{aligned}
$$

which leaves us two parameters Ω_m and C to fit!



Fitting supernova data from the Supernova Cosmology Project, Suzuki et al. 2012

Next step



(with $t_{exp} = t_0 - 1/H_0$)

The models tell us: our universe was initially very dense, crowded — what was that like?