Homogeneous, expanding universes 1

Block Course Introduction to Astronomy and Astrophysics

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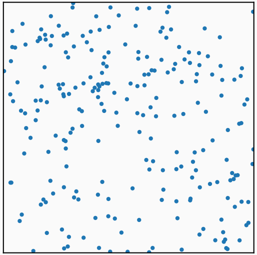
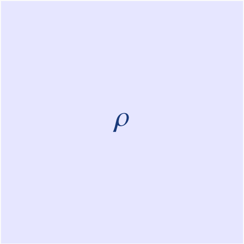
Two complementary views:



“Galaxy dust” with separate, pointlike galaxies



Continuum with uniform density ρ

- Galaxy positions $\vec{x}_i$ in space
- Light can propagate between galaxies

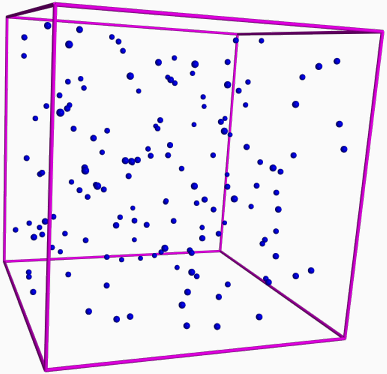
- Change $\rho(t)$ over time
- Derive dynamical equations

Pairwise distances between galaxies i and j

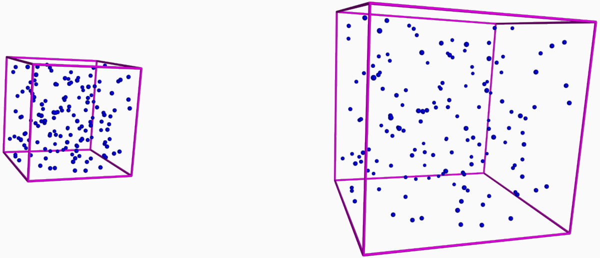
$$d_{ij}(t) = d_{ij}(t_0) = const.$$

Problem: no dynamical mechanism
to stabilize static universe!

(As Einstein 1917 was to find out later)



What change is possible without disturbing isotropy/homogeneity? Simplest possibility: Multiply all pairwise distances with the same factor = scale everything up (or down)

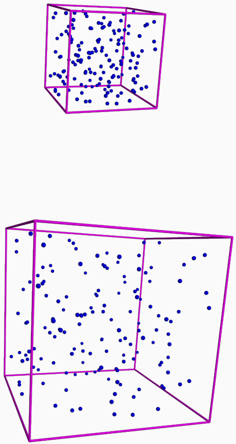


Pattern stays the same – overall scale changes over time

App: `https://astro-apps.org/ExpansionCubes/`

$\Rightarrow$ cosmic expansion with a universal scale factor

- Special family of free-falling galaxies = Hubble flow (=galaxies whose distances change only because of cosmic expansion)
- Choose cosmic time coordinate t corresponding to local time of Hubble-flow galaxies
- In "snapshot" of universe at some time $t = t_0$, introduce coordinates $\vec{x}$ to label Hubble-flow galaxies = comoving coordinates, comoving distances
- Let physical distance between any two Hubble-flow galaxies i, j change as $d_{ij}(t) = a(t) \cdot |\vec{x}_i - \vec{x}_j|$, with $a(t)$ the (universal) cosmic scale factor
- For real galaxies, deviations from Hubble-flow motion are called peculiar velocities



Direct consequence of scale-factor expansion $d_{ij}(t) = a(t)\,|\vec{x}_i - \vec{x}_j| \equiv a(t)\,r_{ij}$:

Define recession speed $v_{ij}(t)$ as change of $d_{ij}(t)$ with cosmic time t:

$$v_{ij}(t) \equiv \frac{\mathrm{d}d_{ij}}{\mathrm{d}t}(t) = \frac{\mathrm{d}a}{\mathrm{d}t}(t) \cdot r_{ij} = \frac{1}{a(t)}\frac{\mathrm{d}a}{\mathrm{d}t}(t) \cdot a(t) \cdot r_{ij} = \frac{1}{a(t)}\frac{\mathrm{d}a}{\mathrm{d}t}(t) \cdot d_{ij}(t)$$

Introduce Hubble parameter $H(t)$ as

$$H(t) \equiv \frac{1}{a(t)}\frac{\mathrm{d}a}{\mathrm{d}t}(t)$$

and we have derived the Hubble-Lemaître-Relation:

$$v_{ij}(t) = H(t) \cdot d_{ij}(t)$$

— valid for all galaxies i, j, with $H(t)$ and $a(t)$ universal functions!

Notation: from now on, short-hand dots for time derivatives, $\dot{a} \equiv \frac{\mathrm{d}a}{\mathrm{d}t}(t)$, $\ddot{a} \equiv \frac{\mathrm{d}^2 a}{\mathrm{d}t^2}(t)$ and similar.

Cosmological redshift as Doppler shift

Cosmological conventions: t_0 as the present time, and introducing the

$$\text{Hubble constant} \quad H_0 \equiv H(t_0) = \left.\frac{\dot{a}}{a}\right|_{t=t_0} .$$

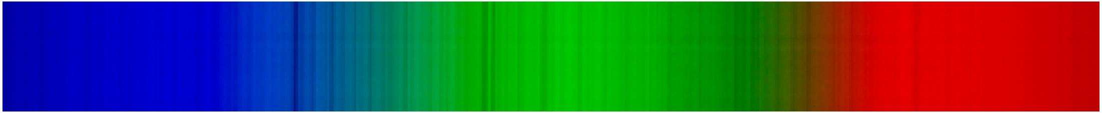
"Local version" of Hubble-Lemaître relation: classical Doppler effect $v = cz$ with

$$z \equiv \frac{\lambda - \lambda_0}{\lambda_0}$$

leads to

$$v = cz = H_0 d$$

where our perspective is Earth-centered: z is the redshift we measure for a galaxy, d the galaxy's distance from Earth, v its recession velocity from Earth



Systematic redshift-distance relation: Hubble-Lemaître relation

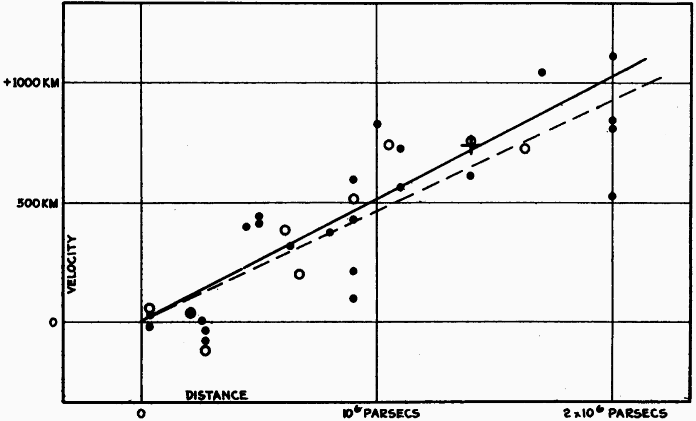
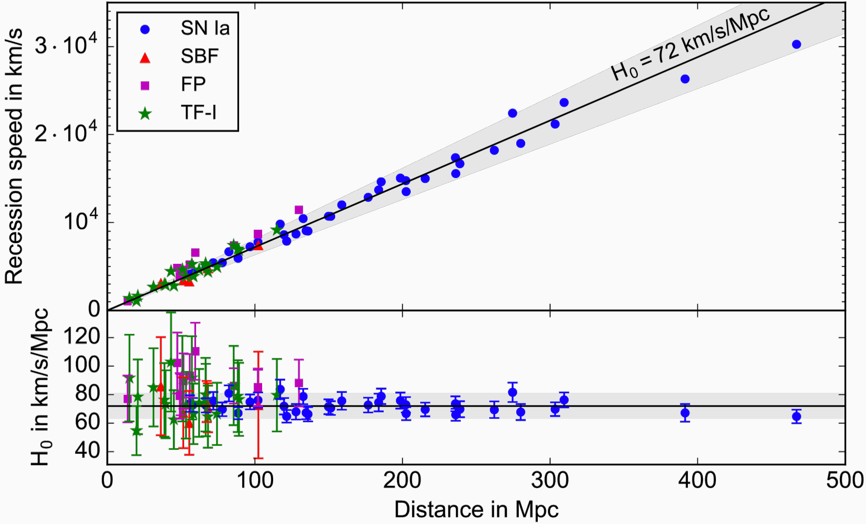


FIGURE 1

Velocity-Distance Relation among Extra-Galactic Nebulae.

Hubble 1929: "A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae" in PNAS 15(3), p. 168ff., for context/predecessors see Trimble 2013

Hubble relation observed: H_0 Key Project 2001



Freedman 2001 et al. (HST Key Project)

Hubble constant value:

$$H_0 \approx 70\,\frac{\text{km/s}}{\text{Mpc}} = 2.3\cdot 10^{-18}\,\text{s}^{-1} = \frac{1}{14\ \text{Gyr}}$$

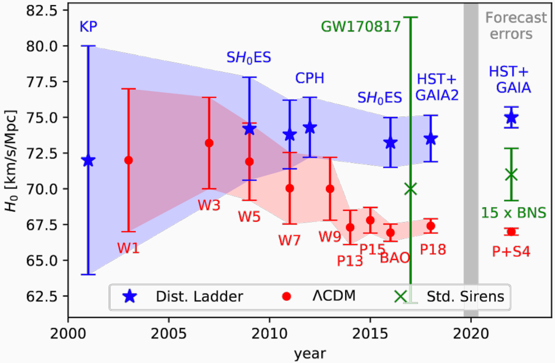
Conventions for keeping your options open:

$$H_0 = h\cdot 100\,\frac{\text{km/s}}{\text{Mpc}} = h_{70}\cdot 70\,\frac{\text{km/s}}{\text{Mpc}}$$

— and keep the h or h_{70} as a variable.

Current crisis in cosmology:
Hubble constant tension!

Planck 67.4 km/s/Mpc vs. 73 km/s/Mpc for Hubble-Lemaître relation measurements



Ezquiaga & Zumalacárregui 2018

Inverse of the Hubble constant is the Hubble time:

$$\tau_H \equiv \frac{1}{H_0} \approx 4 \cdot 10^{17}\,\mathrm{s} = 14\,\mathrm{Gyr}$$

Simple interpretation: if $v = H_0 \cdot d$ is constant, $d/v = 1/H_0$ is time needed to reach distance $d \Rightarrow 1/H_0$ age of the universe. Corresponds to linear expansion $a(t) \sim t$.

c/H_0 is fundamental length scale, Hubble distance

Standard form for Taylor expansion of $a(t)$ is:

$$a(t) \approx a_0 \left[1 + H_0(t - t_0) - \frac{1}{2} H_0^2 q_0 (t - t_0)^2 \right]$$

with the (dimensionless) q_0 the deceleration parameter (oops!), often $a_0 = 1$.

Light-travel time in an expanding universe 1

Light travelling from emitting galaxy e to receiving galaxy r; in snapshot at time t_7, divide connecting line into equal physical distances Δx
with “free-falling waypoints” (e.g. galaxies):

$$\underbrace{t_1 \quad\quad t_2}\underbrace{\quad\quad t_3}\underbrace{\quad\quad t_4}\underbrace{\quad\quad t_5}\underbrace{\quad\quad t_6}\underbrace{\quad\quad t_7}$$

$$t_7 = t_1 + \frac{\Delta x}{c}\cdot\frac{a(t_1)}{a(t_7)} + \frac{\Delta x}{c}\cdot\frac{a(t_2)}{a(t_7)} + \frac{\Delta x}{c}\cdot\frac{a(t_3)}{a(t_7)} + \frac{\Delta x}{c}\cdot\frac{a(t_4)}{a(t_7)} + \frac{\Delta x}{c}\cdot\frac{a(t_5)}{a(t_7)} + \frac{\Delta x}{c}\cdot\frac{a(t_6)}{a(t_7)}$$

Equivalence principle: at galaxy i, cosmic time = local time, physical length corresponding to Δx is $\Delta x \cdot a(t_i)/a(t_7)$, light moves at speed c, so $c\Delta t_i = \Delta x \cdot a(t_i)/a(t_7)$.

Problem: $T \equiv t_7 - t_1$ implicitly depends on the t_i! Re-write each contribution as follows:

$$\Delta t_i = \frac{\Delta x}{c}\cdot\frac{a(t_i)}{a(t_7)} \qquad \Rightarrow \frac{\Delta x}{a(t_7)} = \frac{c\Delta t_i}{a(t_i)}$$

Add up all contributions

$$\frac{\Delta x}{a(t_7)} = \frac{c\Delta t_i}{a(t_i)}$$

to obtain

$$c\sum_{i=1}^{N}\frac{\Delta t_i}{a(t_i)} = \frac{N\cdot\Delta x}{a(t_7)} = \frac{d_{comov}}{a(t_7)}$$

Transition to integrals and infinitesimal sections:

$$\frac{\Delta t_i}{a(t_i)} \approx \int_{t_i}^{t_{i+1}}\frac{\mathrm{d}t}{a(t)} \qquad \text{Integral version:}\quad c\int_{t_e}^{t_r}\frac{\mathrm{d}t}{a(t)} = \frac{d_{comov}}{a(t_0)},$$

with t_e (formerly t_1) emission time and t_r (formerly t_7) the reception time
$\Rightarrow$ implicit equation for travel time!

Cosmological redshift 1

Light leaving distant galaxy at t_e, arriving at ours t_0, comoving distance d_{comov} fulfills

$$c \int_{t_e}^{t_0} \frac{\mathrm{d}t}{a(t)} = \frac{d_{comov}}{a(t_0)}$$

Now consider second light signal: leaving at $t_e + \delta t_e$, arriving at $t_0 + \delta t_0$,

$$\frac{d_{comov}}{a(t_0)} = c \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)}$$

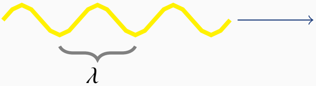
Galaxy pair is the same in each instance, so

$$\int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)} - \int_{t_e}^{t_0} \frac{\mathrm{d}t}{a(t)} = \frac{d_{comov} - d_{comov}}{c \cdot a(t_0)} = 0$$

Re-writing the limits of the integral, using mean-value theorem for integrals:

$$\int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)} - \int_{t_e}^{t_0} \frac{\mathrm{d}t}{a(t)} \approx \frac{\delta t_0}{a(t_0)} - \frac{\delta t_e}{a(t_e)}$$

Apply to light waves with wavelength $\lambda = c \cdot \delta t$:



$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)} \qquad \Rightarrow \qquad 1 + z = \frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)}$$

with λ_e wavelength at emission (local reference frame of distant galaxy),
λ_0 wavelength at reception in our own galaxy: cosmological redshift

Light wavelengths change in the same way as distances between Hubble-flow galaxies!

Caveat: Recession velocity ≠ relative velocity

Caution: Do not naively take recession speeds to be physical speeds!

There is a recipe for comparing velocities in general relativity: parallel transport

Parallel transport from distant Hubble-flow galaxy along light-like geodesic to us gives relativistic radial velocity v_{rel}, with the cosmological redshift given by

$$1 + z = \sqrt{\frac{1 + v_{rel}/c}{1 - v_{rel}/c}} \quad \left(= \frac{a(t_0)}{a(t_e)} \right)$$

— corresponds to special-relativistic formula (Bondi k factor)

Details: Bunn and Hogg 2009, Pössel 2020, arXiv:1912.11677

Redshift drift

$$z(t_r) = \frac{a(t_r)}{a(t_e(t_r))} - 1 \quad \Rightarrow \quad \frac{\mathrm{d}z}{\mathrm{d}t_r} = \frac{\dot{a}(t_r)}{a(t_e(t_r))} - \frac{a(t_r)}{a^2(t_e(t_r))}\dot{a}(t_e(t_r))\frac{\mathrm{d}t_e}{\mathrm{d}t_r}.$$

with t_r reception time, t_e emission time, z cosmological redshift, rewritten:

$$\frac{\mathrm{d}z}{\mathrm{d}t_r} = \frac{a(t_r)}{a(t_e(t_r))}\left[H(t_r) - H(t_e(t_r))\frac{\mathrm{d}t_e}{\mathrm{d}t_r}\right].$$

Use redshift formula for $\mathrm{d}t_e/\mathrm{d}t_r$ and re-write in terms of redshifts and H_0:

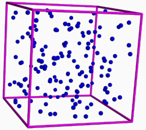
$$H(t_e) = H_0 \cdot (1+z) - \frac{\mathrm{d}z}{\mathrm{d}t_r}$$

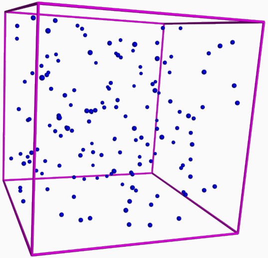
Measuring different z and corresponding $\dot{z}$ allows reconstruction of cosmic history!
$\dot{z} \sim 10^{-10}\,\mathrm{yr}^{-1}$ at $z = 4 \Rightarrow$ ELT ($\sim$ 20 yrs, Liske et al. 2008), or SKA ($\sim$ 12 yrs, Kloeckner et al. 2015)

So far, we have considered effects of (any) scale-factor expansion $a(t)$:

- Hubble-Lemaître relation
- Cosmological redshift

Next: How do densities change? What is the dynamics that determines $a(t)$?





In the continuum picture: $a(t)$ is universal
$\Rightarrow$ we can study it on smallest scales, where classical (Newtonian) physics is valid

Consider small co-moving cube. Energy conservation reads:

$$\mathrm{d}U = \delta Q - p\,\mathrm{d}V$$

Since no heat “enters universe from the outside”, $\delta Q = 0$.

Mass density corresponds to energy $U = \rho c^2 V$. Rewrite as

$$\mathrm{d}\rho = -(\rho + p/c^2)\frac{\mathrm{d}V}{V}$$

Volume changes as

$$V(t) = \left(\frac{a(t)}{a(t_0)}\right)^3 V_0 \quad \Rightarrow \quad \frac{\mathrm{d}V}{V} = 3\frac{\mathrm{d}a}{a} \quad \Rightarrow \quad \mathrm{d}\rho = -3(\rho + p/c^2)\frac{\mathrm{d}a}{a}$$

How densities change over time:

$$\dot{\rho} = -3(\rho + p/c^2)\frac{\dot{a}}{a}$$

…depends on equation of state (EOS), $p = p(\rho)$.

Important EOSs in cosmology:

Matter ("galaxy dust" or dark matter)	$p = 0$	$w = 0$
Electromagnetic radiation	$p = \rho c^2/3$	$w = 1/3$
Dark energy (cosmological constant)	$p = -\rho c^2$	$w = -1$

with w defined by $p = w \cdot \rho c^2$.

How densities change over time:

$$\mathrm{d}\rho = -3(\rho + p/c^2)\frac{\mathrm{d}a}{a} \quad \text{with} \quad p = w \cdot \rho c^2$$

$$\Rightarrow \quad \frac{\mathrm{d}\rho}{\rho} = -3(1+w)\,\frac{\mathrm{d}a}{a}$$

$$\Rightarrow \quad \rho(t) = \rho(t_0) \cdot \left(\frac{a(t)}{a(t_0)}\right)^{-3(1+w)}$$

Simple case: Galaxy dust (matter) or dark matter has, as expected (constant number of particles plus $V \sim a^3$):

$$\frac{\rho_M(t)}{\rho_M(t_0)} = \left(\frac{a(t)}{a(t_0)}\right)^{-3}$$

For dark energy, $1 + w = 0$ together with

$$\rho(t) = \rho(t_0) \cdot \left(\frac{a(t)}{a(t_0)}\right)^{-3(1+w)}$$

leads to constant, time-independent density:

$$\rho(t) = \rho(t_0)$$

Meshes with original introduction by Einstein 1917 of cosmological constant
(there: to stabilize a static universe)

144 Sitzung der physikalisch-mathematischen Klasse vom 8. Februar 1917

der an sich nicht beansprucht, ernst genommen zu werden; er dient nur dazu, das Folgende besser hervortreten zu lassen. An die Stelle der Poissonschen Gleichung setzen wir

$$\Delta\phi - \lambda\phi = 4\pi K\rho\,, \qquad (2)$$

wobei λ eine universelle Konstante bedeutet. Ist ρ_0 die (gleichmäßige) Dichte einer Massenverteilung, so ist

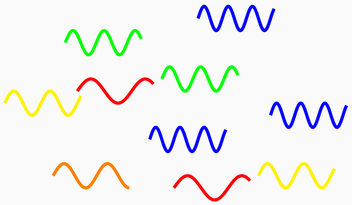
$$\phi = -\frac{4\pi K}{\lambda}\rho_0 \qquad (3)$$

eine Lösung der Gleichung (2). Diese Lösung entspräche dem Falle, daß die Materie der Fixsterne gleichmäßig über den Raum verteilt wäre, wobei die Dichte ρ_0 gleich der tatsächlichen mittleren Dichte der Materie des Weltraumes sein möge. Die Lösung entspricht einer unendlichen Ausdehnung des im Mittel gleichmäßig mit Materie erfüllten Raumes. Denkt man sich, ohne an der mittleren Verteilungsdichte etwas zu ändern, die Materie örtlich ungleichmäßig verteilt, so wird sich über den konstanten ϕ-Wert der Gleichung (3) ein zusätzliches ϕ überlagern, welches in der Nähe dichterer Massen einem Newtonschen Felde um so ähnlicher ist, je kleiner λ, gegenüber $4\pi K\rho$ ist.

Einstein 1917 — English translation available

More interesting: electromagnetic radiation!

- photon gas
- photons travelling in straight lines
- photon number conserved

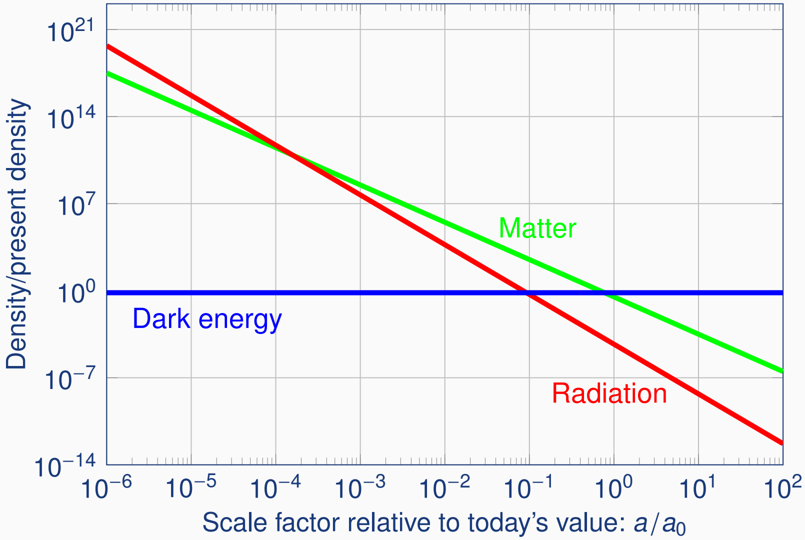


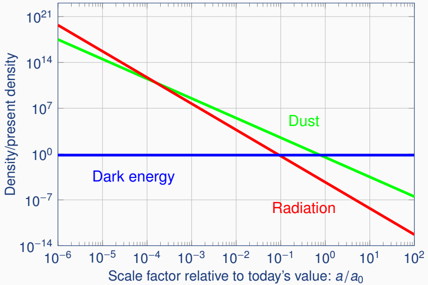
$$\frac{\rho_R(t)}{\rho_R(t_0)} = \left(\frac{a(t)}{a(t_0)}\right)^{-4} = \left(\frac{a(t)}{a(t_0)}\right)^{-3} \cdot \left(\frac{a(t)}{a(t_0)}\right)^{-1}$$

which, with single photon energy $E = h\nu = hc/\lambda$, meshes with cosmological redshift,

$$\lambda(t) = \lambda_0 \cdot \frac{a(t)}{a(t_0)}$$

Different eras depending on the scale factor





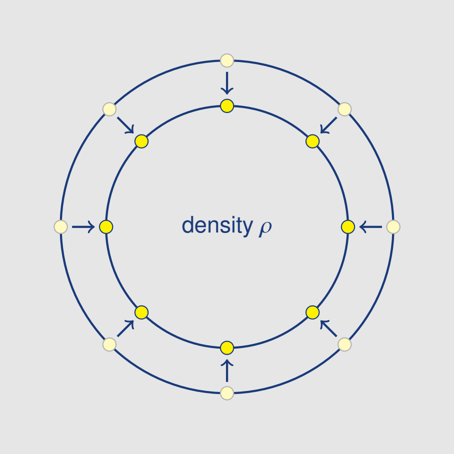
Two caveats:

- This says little about evolution — some values of a might not even be reached!
- In reality, matter will change — particles might start as dust (non-relativistic) and, at smaller a, end up at high energies and thus as radiation (relativistic particles)

Simplified form of Einstein's equations for sphere of free-fall particles = Hubble-flow galaxies:

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -4\pi G \cdot \left[\rho + \frac{3p}{c^2}\right]$$

with $\rho(t)$ the universe's matter density; contains energy contributions as per $E = mc^2$; general relativity adds pressure term to Newtonian description



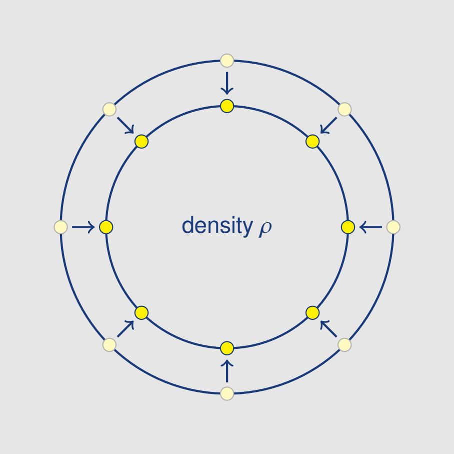
Spherical volume: $V \sim d(t)^3$, so that

$$\frac{\ddot{V}}{V} = \frac{1}{d^3(t)} \cdot \frac{\mathrm{d}^2(d^3)}{\mathrm{d}t^2} = 3\frac{\ddot{d}}{d}$$

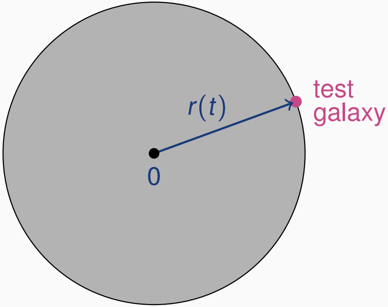
Since $d(t) = a(t) \cdot r_{comov}$:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho + \frac{3p}{c^2}\right]$$

Second-order Friedmann equation



Except for pressure term, this is the Newtonian result!



Derivation for how distance $r(t)$ for other galaxy, with our own galaxy at the origin, changes over time:

Newton's shell argument: only inner mass contributes, as if concentrated at $r = 0$:

$$\ddot{r} = -\frac{GM}{r^2}.$$

- $M = V \cdot \rho$ (Newtonian)
- $V = 4/3 \cdot \pi \cdot r(t)^3$
- $r(t) = a(t)/a(t_0) \cdot r_0$

Specific r_0 and $a(t_0)$ drop out,

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\rho$$

First-order Friedmann equation

Substituting our density equation (from small-scale energy conservation),

$$\dot{\rho} = -3(\rho + p/c^2)\frac{\dot{a}}{a}$$

allows second-order Friedmann equation to be integrated, giving

first(-order) Friedmann equation:

$$\frac{\dot{a}^2 + Kc^2}{a^2} = \frac{8\pi G}{3}\rho$$

where K is an integration constant.

Separate K into sign k and magnitude $1/R_0^2$: $K = k/R_0^2$, so that $k = 0, -1, +1$.